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Electromagnetic Wave Propagation in Relativistic Magnetized Plasmas

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An improved mathematical technique and a new code for deriving the conductivity tensor for collisionless plasmas have been developed. The method is applicable to a very general case, including both hot (relativistic) and cold magnetized plasmas, with only isotropic equilibrium distributions being considered here. The usual derivation starts from the relativistic Vlasov equation and leads to an integration over an infinite sum of Bessel functions which has to be done numerically. In the new solution the integration is carried out over a product of two Bessel functions only. This reduces the computing time very significantly. An added advantage over existing codes is our capability to perform the computations for waves propagating obliquely to the magnetic field. Both improvements greatly facilitate investigations of properties of the plasma under conditions hitherto unexplored.

I. INTRODUCTION

The propagation of electromagnetic waves in plasmas is a fundamental theoretical problem of plasma physics. It also has important applications in most experimental fusion facilities.

Relativistic temperatures have been reached, for example, in the electron rings of bumpy torus devices, such as the EBT at the Oak Ridge National Laboratory, and several mirror machines. The crucial stage in the analysis of this problem is the calculation of the conductivity tensor. In this paper we present an improved method of calculating this tensor in the general temperature range, including hot (relativistic) magnetized plasmas. This method not only increases the efficiency of the calculation substantially (a factor of 100 in computer time in some cases compared to previous methods), but also enables us to investigate domains of parameters (temperatures, wave numbers, etc.) that have not been explored before, such as hot plasmas with $k_{\|} \neq 0$, i.e., waves propagating at an oblique angle to the magnetic field.

To solve the wave propagation problem one can start from the linearized Vlasov equation, which can be taken generally as relativistic. The usual solution leads to an infinite sum of terms, each of which contains a Bessel function of the momentum and has to be integrated over the momentum space. An analytic approximation of the result is possible only in special cases, such as cold plasmas (see e.g., Krall and

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Trivelpiece [1], Stix [2], Sitenko and Stepanov [3]) or warm plasma near the cyclotron resonance (I. P. Shkarofsky [4], K. Imre and W. Weitzner [5]). In the general case the integration has to be done numerically. Calculations of this kind have been carried out (Batchelor and Goldfinger [6]) and have proven to be very costly in computer time terms. The new method is based on a new solution of the Vlasov equation, introduced in this paper, which leads to a much simpler result, i.e., an integral over one term involving two Bessel functions only.

The numerical integration has presented us with several problems. Some of the techniques for their solution are briefly described below:

1. An infinite number of singularities arising from a factor $f(\omega)/\sin(\pi\omega)$ in the integrand. We subtracted and added a term $f(n)/\sin(\pi\omega)$ where n is the singular point (an integer). This makes the integral regular, while the added term is easier to handle then the original integrand.

2. The integration is over a two dimensional semi-infinite domain with a weight function decaying very slowly (at high temperatures) towards infinity. We used the Gauss–Laguerre integration method on the semi-infinite variable and the Gauss–Legendre method on the other variable.

3. Routines for calculating the Bessel functions $J_\nu(z)$ were available on the MFE network only with the restriction of a real argument. For a complex argument we have expanded the functions in a Taylor series and extended the expansion to the complex domain. To avoid the singularities near $z=0$, arising from the factor z^ν at the beginning of the power series, we have expanded the function $z^{-\nu}J_\nu(z)$ rather than $J_\nu(z)$ and divided the result by $z^{-\nu}$.

4. We tried several methods to find the zeroes of the dispersion function. A second order Newton–Raphson method (in the complex plane) with back averaging to help dampen the oscillations was the most successful over most of the temperature range. Some ready-made zero finders, like the ZANLYT of the IMSL library and the SNSQE of the SLATEC library, were also good at low and intermediate temperatures (except at $\mathbf{k}\approx 0$). At high temperatures the solution may sometimes have to be found graphically, as the dispersion function behaves less well in this region.

The next section will describe in detail the mathematical solution of the relativistic Vlasov equation. Section III will describe the integration methods and in Section IV we shall give and discuss various results.

II. Derivation of the Conductivity Tensor

The relativistic Vlasov equation in the Fourier space is [7]

$$i(\omega+\mathbf{k}\cdot\mathbf{v})f+\Omega_c\mathbf{v}\times\hat{\mathbf{B}}_0\cdot\frac{\partial f}{\partial\mathbf{p}}=-\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}_1\right)\cdot\frac{\partial G}{\partial\mathbf{p}}$$

where

$$\mathbf{p} = \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} \qquad \mathbf{v} = \frac{\mathbf{p}}{\sqrt{1 + p^2/c^2}}$$

$$\gamma(p) = \sqrt{1 + p^2/c^2} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and $G = G(\mathbf{p})$ is the equilibrium function. In the following we shall restrict ourselves to an isotropic G, and thus the terms proportional to the magnetic perturbation $\mathbf{B}_1$ will vanish. For a G depending on the pitch angle, these terms would be retained throughout the following treatment, with $\mathbf{B}_1$ being eliminated by the Maxwell equation $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}_1$. These terms can be handled, with no particular difficulty, by the same procedure that we shall apply to the remaining ones.

We choose our coordinate system as follows:

$$\mathbf{B}_0 = (0, 0, 1)$$

$$\mathbf{k} = (0, k_\perp, k_\parallel)$$

$$\mathbf{v} = (v_\perp \cos\phi, v_\perp \sin\phi, v_\parallel)$$

$$\mathbf{p} = (p_\perp \cos\phi, p_\perp \sin\phi, p_\parallel).$$

In this coordinate system we can develop some of the terms of the equation as

$$\mathbf{k} \cdot \mathbf{v} = k_\perp v_\perp \sin\phi + k_\parallel v_\parallel$$

$$\mathbf{v} \times \hat{\mathbf{B}}_0 \cdot \frac{\partial f}{\partial \mathbf{p}} = \frac{1}{\gamma(p)} \mathbf{p} \times \hat{\mathbf{B}}_0 \cdot \frac{\partial f}{\partial \mathbf{p}} = -\frac{1}{\gamma(p)} \frac{\partial f}{\partial \phi},$$

and obtain a linear first order differential equation for $f(\phi)$:

$$i(\omega\gamma(p) + k_\parallel p_\parallel + k_\perp p_\perp \sin\phi) f - \Omega_c \frac{\partial f}{\partial \phi} = -\frac{e}{m} \gamma(p) \mathbf{E} \cdot \mathbf{p} \frac{\partial G}{\partial p^2/2}.$$

Going over to the circularly polarized electric vector $\mathbf{E}$ one has

$$\begin{aligned} \mathbf{E} \cdot \mathbf{p} &= E_x p_\perp \cos\phi + E_y p_\perp \sin\phi + E_z p_\parallel \\ &= p_\perp e^{i\phi} \cdot \tfrac{1}{2}(E_x - iE_y) + p_\perp e^{-i\phi} \cdot \tfrac{1}{2}(E_x + iE_y) + E_z p_\parallel \\ &= p_\perp (e^{i\phi} E_- + e^{-i\phi} E_+) + E_\parallel p_\parallel. \end{aligned}$$

We are interested in finding the current vector $\mathbf{J}$ defined as

$$\mathbf{J} = \rho \int f(p) \mathbf{v} \, d^3p$$

or

$$J_{\pm} = \rho \int f \cdot \frac{1}{2}(v_x \pm i v_y)\, d^3p$$

$$J_{\parallel} = \rho \int f v_z\, d^3p.$$

Using the previous relations these can be written as

$$J_{\pm} = \tfrac{1}{2}\rho \int \frac{p_{\perp}}{\gamma(p)} e^{\pm i\phi} f\, d^3p = \tfrac{1}{2}\rho \int \frac{f}{\gamma(p)} e^{\pm i\phi} p_{\perp}^2\, dp_{\perp}\, dp_{\parallel}\, d\phi$$

$$J_{\parallel} = \rho \int \frac{f}{\gamma(p)} p_{\perp} p_{\parallel}\, dp_{\perp}\, dp_{\parallel}\, d\phi.$$

Thus one has to find f and substitute it in **J**. The following solution satisfies the differential equation for $f(\phi)$, namely, the relativistic Vlasov equation, as well as the condition of periodicity in ϕ,

$$f = \frac{\exp i(\omega^*\phi - k_{\perp} p_{\perp} \cos\phi)/\Omega_c}{e^{-2\pi i\omega^*/\Omega_c} - 1} \int_{\phi}^{\phi+2\pi} d\phi' \exp - i(\omega^*\phi' - k_{\perp} p_{\perp} \cos\phi')/\Omega_c$$

$$\times \frac{e}{m\Omega_c} [(E_- e^{i\phi'} + E_+ e^{-i\phi'})\, p_{\perp} + E_{\parallel} p_{\parallel}] \frac{\partial G}{\partial p^2/2} \gamma(p)$$

where $\omega^* = \omega\gamma(p) + k_{\parallel} p_{\parallel}$. For simplicity we renormalize the variables as

$$\frac{\omega}{\Omega_c} = \bar{\omega}, \quad \frac{k_{\parallel} c}{\Omega_c} = \bar{k}_{\parallel}, \quad \frac{k_{\perp} c}{\Omega_c} = \bar{k}_{\perp},$$

$$\frac{e}{m\Omega_c} E_{\pm} = \bar{E}_{\pm}, \quad \frac{e}{m\Omega_c} E_{\parallel} = \bar{E}_{\parallel}, \quad \frac{\mathbf{p}}{c} = \bar{\mathbf{p}}.$$

In the expression for f we set $\phi' = \phi + \phi''$ to obtain (dropping the bars)

$$f = \frac{\gamma(p)\, \partial G/\partial(p^2/2)}{e^{-2\pi i\omega^*} - 1} \int_0^{2\pi} d\phi'' \exp[-i\omega^*\phi'' - 2ik_{\perp} p_{\perp} \sin(\phi + \phi''/2) \sin(\phi''/2)]$$

$$\times [p_{\perp}(E_+ e^{-i\phi} e^{-i\phi''} + E_- e^{i\phi} e^{i\phi''}) + E_{\parallel} p_{\parallel}].$$

The current $\mathbf{J}'$ is now (denoting $J'_{\pm} = 2J_{\pm}$, $J'_{\parallel} = J_{\parallel}$)

$$\begin{pmatrix} J'_+ \\ J'_- \\ J'_\| \end{pmatrix} = \rho \int \frac{p_\perp \, dp_\perp \, dp_\| \, G'}{e^{-2\pi i\omega^*} - 1} \int d\phi \int_0^{2\pi} d\phi''$$

$$\times \exp[-i\omega^* \phi'' - 2ik_\perp p_\perp \sin(\phi + \phi''/2) \sin(\phi''/2)]$$

$$\times \begin{pmatrix} p_\perp^2 e^{-i\phi''} & p_\perp^2 e^{2i\phi} e^{i\phi''} & p_\perp p_\| e^{i\phi} \\ p_\perp^2 e^{-2i\phi - i\phi''} & p_\perp^2 e^{i\phi''} & p_\perp p_\| e^{-i\phi} \\ p_\perp p_\| e^{-i\phi - i\phi''} & p_\perp p_\| e^{i(\phi + \phi'')} & p_\|^2 \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \\ E_\| \end{pmatrix}$$

where

$$G' = \frac{\partial G(p)}{\partial p^2/2}.$$

We make the change of variables

$$\bar{\phi} = \phi + \phi''/2$$
$$\tilde{\phi} = \phi''/2 - \pi/2,$$

to obtain

$$(\mathbf{J}') = 2\rho \int \frac{p_\perp \, dp_\perp \, dp_\| \, G' e^{-\pi i\omega^*}}{e^{-2\pi i\omega^*} - 1} \int d\bar{\phi} \int_{-\pi/2}^{\pi/2} d\tilde{\phi}$$

$$\times \exp[-2i\omega^* \tilde{\phi} - 2ik_\perp p_\perp \sin \bar{\phi} \cos \tilde{\phi}]$$

$$\times \begin{pmatrix} -p_\perp^2 e^{-2i\tilde{\phi}} & p_\perp^2 e^{2i\tilde{\phi}} & -ip_\perp p_\| e^{i\bar{\phi}} e^{-i\tilde{\phi}} \\ p_\perp^2 e^{-2i\tilde{\phi}} & -p_\perp^2 e^{2i\tilde{\phi}} & ip_\perp p_\| e^{-i\bar{\phi}} e^{i\tilde{\phi}} \\ -ip_\| p_\perp e^{-i\bar{\phi}} e^{-i\tilde{\phi}} & ip_\| p_\perp e^{i\bar{\phi}} e^{i\tilde{\phi}} & p_\|^2 \end{pmatrix} (\mathbf{E}).$$

To do the $\bar{\phi}$ integration we use the formulae [8]

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi + c}^{\pi + c} e^{iz \sin \alpha} e^{in\alpha} \, d\alpha$$

$$J_{-n} = (-1)^n J_n$$

where $J_n(z)$ are Bessel functions (c is an arbitrary constant) and obtain

$$(\mathbf{J}') = 2\pi i\rho \int \frac{p_\perp \, dp_\perp \, dp_\| \, G'}{\sin \pi\omega^*} \int_{-\pi/2}^{\pi/2} e^{-2i\tilde{\phi}\omega^*} \, d\tilde{\phi}$$

$$\times \begin{pmatrix} -p_\perp^2 e^{-2i\tilde{\phi}} J_0(\xi) & p_\perp^2 J_2(\xi) & ip_\perp p_\| e^{-i\tilde{\phi}} J_1(\xi) \\ p_\perp^2 J_2(\xi) & -p_\perp^2 e^{2i\tilde{\phi}} J_0(\xi) & ip_\perp p_\| e^{i\tilde{\phi}} J_1(\xi) \\ -ip_\| p_\perp e^{i\tilde{\phi}} J_1(\xi) & -ip_\perp p_\| e^{i\tilde{\phi}} J_1(\xi) & p_\|^2 J_0(\xi) \end{pmatrix} (\mathbf{E}).$$

where

$$\xi = 2k_\perp p_\perp \cos\tilde{\phi}.$$

We now combine the two halves of each integral, i.e., the part on the interval $[-\pi/2, 0]$ with the part on $[0, \pi/2]$, into one integral over $[0, \pi/2]$ and obtain

$$\mathbf{J}' = 4\pi i\rho \int \frac{p_\perp\, dp_\perp\, dp_\parallel\, G'}{\sin \pi\omega^*} \int_0^{\pi/2} d\tilde{\phi}\, \ddot{\mathbf{M}}' \cdot \mathbf{E}$$

with

$$\ddot{\mathbf{M}}' = \begin{pmatrix} -p_\perp^2 J_0 \cos 2(\omega^*+1)\tilde{\phi} & p_\perp^2 J_2 \cos 2\omega^*\tilde{\phi} & ip_\perp p_\parallel J_1 \cos(2\omega^*+1)\tilde{\phi} \\ p_\perp^2 J_2 \cos 2\omega\tilde{\phi} & -p_\perp^2 J_0 \cos 2(\omega^*-1)\tilde{\phi} & ip_\perp p_\parallel J_1 \cos(2\omega^*-1)\tilde{\phi} \\ -ip_\perp p_\parallel J_1 \cos(2\omega^*+1)\tilde{\phi} & -ip_\perp p_\parallel J_1 \cos(2\omega^*-1)\tilde{\phi} & p_\parallel^2 J_0 \cos 2\omega^*\tilde{\phi} \end{pmatrix}$$

where $J_n = J_n(\xi)$.

We can now use the formula

$$J_\mu(z)\, J_\nu(z) = \frac{2}{\pi} \int_0^{\pi/2} J_{\nu+\mu}(2z\cos\theta) \cos(\mu-\nu)\,\theta\, d\theta.$$

We can identify z and θ in the formula above with our variables $k_\perp p_\perp$ and $\tilde{\phi}$. To bring the integral on each element in $\ddot{\mathbf{M}}'$ to the form of the right hand side of the above formula, we identify the factor of $\tilde{\phi}$ in the argument of each cosine function with $\mu - \nu$, and the index of each $J_{0,1,2}$ with $\mu + \nu$, so that we obtain a simple set of two equations for μ, ν, for each element. We finally obtain

$$\mathbf{J}' = 2\pi^2 i\rho \int \frac{p_\perp\, dp_\perp\, dp_\parallel\, G'}{\sin \pi\omega^*} \ddot{\mathbf{M}} \cdot \mathbf{E}$$

with

$$\ddot{\mathbf{M}} = \begin{pmatrix} -p_\perp^2 J_{\omega^*+1} J_{-\omega^*-1} & p_\perp^2 J_{\omega^*+1} J_{-\omega^*+1} & ip_\perp p_\parallel J_{-\omega^*} J_{\omega^*+1} \\ p_\perp^2 J_{\omega^*+1} J_{-\omega^*+1} & -p_\perp^2 J_{\omega^*-1} J_{-\omega^*+1} & ip_\perp p_\parallel J_{\omega^*} J_{-\omega^*+1} \\ -ip_\perp p_\parallel J_{-\omega^*} J_{\omega^*+1} & -ip_\perp p_\parallel J_{\omega^*} J_{-\omega^*+1} & p_\parallel^2 J_{\omega^*} J_{-\omega^*} \end{pmatrix}$$

where

$$J_\nu = J_\nu(z) = J_\nu(k_\perp p_\perp).$$

Thus, we have obtained an integrand which is a product of two Bessel functions only. The integration variables $p_\perp$ and $p_\parallel$ now appear both in the argument of the Bessel functions and in their index.

The usual infinite-series solution of the Vlasov equation is the Mittag–Lefler expansion [9] of the above solution.

We note that in going back to the original normalization we only have to replace the factor $2\pi^2$ in front of the integral by $\pi\omega_p^2/2\Omega_c$ where ω_p is the plasma frequency $\omega_p^2 = 4\pi ne^2/m$.

It now remains to integrate each term over the momentum variables $p_\perp$, $p_\parallel$. The resulting conductivity tensor $\ddot{\sigma}$, defined by $\mathbf{J} = \ddot{\sigma}\mathbf{E}$, is then inserted in the dispersion function D

$$D = \det[\ddot{\mathbf{I}}(1 - n^2) + \mathbf{nn} + \frac{4\pi i}{\omega}\ddot{\sigma}]$$

whose roots have to be found ($\mathbf{n} \equiv \mathbf{k}c/\omega$).

A representation of D employing a product of two Bessel functions has been previously obtained for the special case of an electrostatic, nonrelativistic plasma having a Lorentzian distribution in the parallel direction, by Hall, Heckrotte, and Kammash [10].

III. The Integration Method

The integrand derived in the last section is of the form $f(\omega^*)/\sin(\pi\omega^*)$ (where $\omega^* = \omega^*(p_\perp, p_\parallel)$) and thus it has a singularity at every integer ω^*. Although these singularities do not lead to overflows in the computer, they make the integral indefinite; i.e., it does not converge to any definite value when refining the mesh.

To handle the situation we subtract and add a term $f(n)/\sin(\pi\omega^*)$ in the interval around each singularity $\omega^* = n$ (n = integer). Thus we can write

$$\int_{n-1/2}^{n+1/2} \frac{f(\omega^*)}{\sin \pi\omega^*}\, d\omega^* = \int_{n-1/2}^{n+1/2} \frac{f(\omega^*) - f(n)}{\sin \pi\omega^*}\, d\omega^* + \int_{n-1/2}^{n+1/2} \frac{f(n)}{\sin \pi\omega^*}\, d\omega^*.$$

The first integral on the right hand side is regular and poses no problem. The second integral has a constant numerator and is thus easier to deal with. It is convenient to use ω^* as one of the integration variables (rather then $p_\perp$ or $p_\parallel$). We divide the range of integration over ω^* into symmetric intervals around the integers and use the Plemelj formula in each interval. The singular integral in the above formula will read

$$\int_{n-1/2}^{n+1/2} \frac{f(n)}{\sin \pi\omega^*}\, d\omega^* = P \int \frac{f(n)}{\sin \pi\omega^*}\, d\omega^* - i\delta(\omega^* - n)(-1)^n.$$

The principal value vanishes due to the symmetry of the interval (except at the beginning of the integration range) and we are left only with a δ function.

A variation of this method can be multiplying the constant $f(n)$ by a symmetric function $g(\omega^*)$ with $g(n) = 1$ (on the singularities), for example, $g = \cos^2(\pi\omega^*)$. This

can be used to make the final integrand smoother at the borders between the integration intervals $(n \pm \frac{1}{2})$ so that the numerical integration is more accurate.

The other coordinate in the two-dimensional integration has to be chosen in such a way that the Jacobian of the transformation from $p_\perp$, $p_\parallel$ to the new coordinates will not be singular. Neither $p_\perp$ nor $p_\parallel$ will do and we have found the resulting singularities difficult to remove. Thus we use the quantity $\bar{p} = p_\parallel/\gamma(p)$ as a second coordinate. It has the range $[-1, +1]$ and corresponds to an angle in a polar coordinate system.

The inverse transformation is (denoting $\bar{\omega} = \omega^*/\omega$, $\bar{p} = p_\parallel/\gamma(p)$, and $n_\parallel = k_\parallel c/\omega$)

$$p_\parallel = \frac{\bar{\omega}\bar{p}}{1 + n_\parallel \bar{p}}$$

$$p_\perp = \left[\frac{(1-\bar{p}^2)\,\bar{\omega}^2}{(1+n_\parallel \bar{p})^2} - 1\right]^{1/2}$$

while the Jacobian is

$$\frac{\partial(p_\perp, p_\parallel)}{\partial(\omega^*, \bar{p})} = -\frac{\gamma(p)^2}{(1+n_\parallel \bar{p})\, p_\perp \omega}.$$

It is obvious from the expression for $p_\perp$ that $|\bar{\omega}|$ has to be greater than a minimal value $\bar{\omega}_{\min}$ for $p_\perp$ to be real:

$$\bar{\omega}_{\min} = \frac{1 + n_\parallel \bar{p}}{\sqrt{1-\bar{p}^2}}.$$

It is convenient to substitute this value in the expression for $p_\perp$ to obtain

$$p_\perp = [(1-\bar{p}^2)(\Delta\bar{\omega}^2 + 2\Delta\bar{\omega}\bar{\omega}_{\min})]^{1/2} \frac{1}{1+n_\parallel \bar{p}},$$

where $\Delta\bar{\omega} = \bar{\omega} - \bar{\omega}_{\min}$. For $n_\parallel = 0$ one can see from the expression for $p_\perp$ that the points $(\omega^*, \bar{p})$ and $(\omega^*, -\bar{p})$ correspond to the same $p_\perp$. As the Bessel functions depend only on ω^* and $p_\perp$, one can save on the number of calls to the expensive Bessel subroutines by storing their values computed for the positive $\bar{p}$ mesh points and use them for the points with negative $\bar{p}$. For $n_\parallel \neq 0$ (but $\ll 1$) the situation is a little more complicated. We divide the interval $[-1, 1]$ in two: $a = [-1, -n_\parallel]$ and $b = [-n_\parallel, 1]$. For every point $\bar{p} \in b$ there is a point $\bar{p}' \in a$ corresponding to the same $p_\perp$ (and the same ω^*) and thus having the same Bessel function. This point is

$$\bar{p}' = \frac{s}{1 - n_\parallel s}$$

where

$$s = \frac{1}{n_\parallel^2 - 1} \frac{1}{1 + n_\parallel \bar{p}} [2n_\parallel + \bar{p}(n_\parallel^2 + 1)].$$

In the interval b we integrate over $\bar{p}$ as before, and in the interval a we integrate over $\bar{p}'$. The Jacobian of the transformation from $\bar{p}$ to $\bar{p}'$ is

$$\frac{d\bar{p}}{d\bar{p}'} = -(1 - n_\parallel s)^2 (1 + n_\parallel \bar{p})^2.$$

The equilibrium function $G(p)$ used in the integrand was the relativistic Maxwellian,

$$G(p) = \frac{\alpha}{4\pi k_2(\alpha)} e^{-\alpha\gamma(p)},$$

where $\alpha = m_0 c^2/KT$ and $k_2(\alpha)$ is the Bessel function of the third kind. In our coordinates we have

$$\gamma(p) = \frac{\bar{\omega}}{1 + n_\parallel \bar{p}}.$$

Thus for a fixed $\bar{p}$ we have approximately an exponential weight function in the variable ω^*, decaying quite slowly at high temperatures, and leaving a long tail to integrate upon. We have found the Gauss–Laguerre integration method [11] most suitable for this situation. The integral is replaced by a sum

$$\int_a^\infty e^{-bx} f(x)\, dx = \sum_{i=1}^{n} W_i f(x_i)$$

where

$$x_i = \frac{x_i'}{b} + a$$

$$W_i = \frac{W_i'}{b}.$$

x_i' are the zeroes of an nth order Laguerre polynomial and W_i are the weights for $b = 1$; n is the number of mesh points. We take the x_i', W_i' from a NAG library routine and modify them using the above formulae.

For the $\bar{p}$-integration we use the Gauss–Legendre integration method, which is suitable for a finite interval.

IV. Results and Conclusions

The code has been run and tested on a CRAY-1 computer on the MFE network. Some of the conclusions are summarized below.

1. *Comparison with Other Results*

Other work has been done only for limited subsets of the parameter space in which our code is applicable. For example, there is no analytical or numerical work for $k_{\|} \neq 0$, except at low temperatures near the cyclotron resonance. Using our code with $k_{\|} \neq 0$ enables us to investigate waves propagating obliquely in respect to the magnetic field. Regarding temperatures, we have tried the range from 0.1 keV to well over 1 MeV and did not encounter any particular problem. We have compared our program with the much slower one of Batchelor and Goldfinger at $k_{\|} = 0$, for a wide range of temperatures and frequencies, and found generally a very good agreement (Figs. 1, 2). At low temperatures we could compare it with Imre's work, with $k_{\|} \neq 0$, and with the analytical expansion of Shkarofsky, again with a very good agreement (Fig. 2).

A comparison with the nonrelativistic analytic results of Sitenko and Stepanov has been made, after modifying our code to a nonrelativistic one. The agreement below 20 keV (at $k_{\perp} = 0.5$) was about 0.1%. (In their expansion $Tk_{\perp}^2 \ll 1$.)

2. *Timing*

At 50 keV we needed about 0.05 CRAY seconds to compute the conductivity tensor. At 300 keV the time rose to 0.12 s. Finding a root of the dispersion function requires less than this time multiplied by the number of iterations, because when

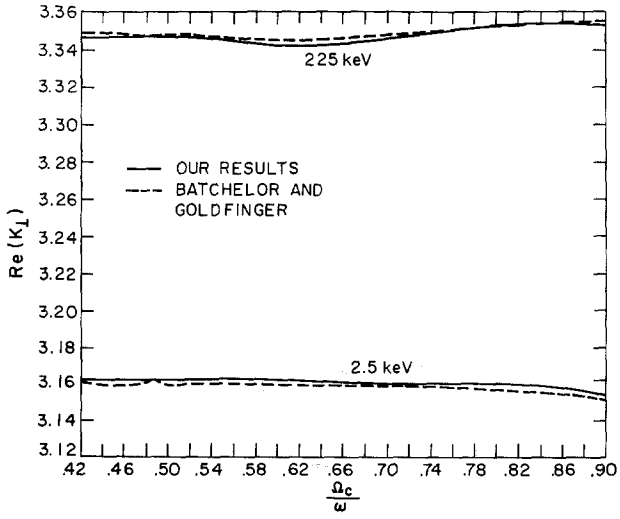

FIG. 1. Comparison between our results and Batchelor and Goldfinger's (ordinary mode).

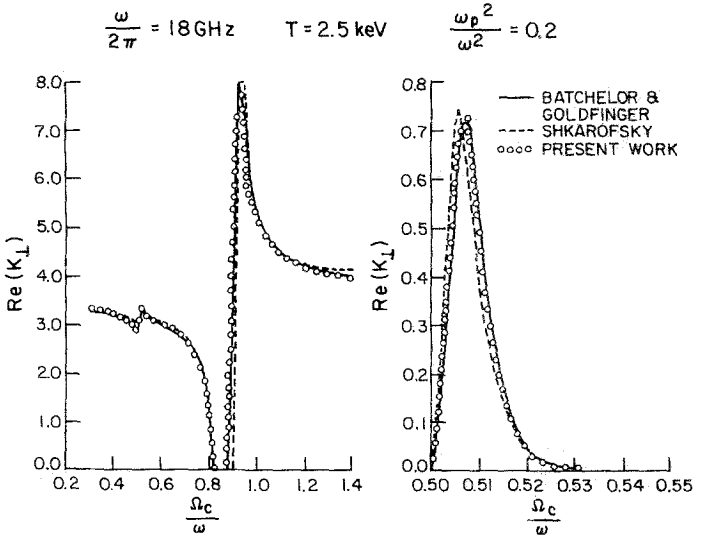

FIG. 2. Coparison between fully relativistic dispersion code and weakly relativistic expansion (Shkarofsky) (extraordinary mode).

the iterations are close, we can use the Taylor expansion of the Bessel functions already existing in the code (for calculation with complex argument) rather than re-calculating the Bessel functions. For a good initial guess, the program does 8–10 iterations in about twice the time of one. For an arbitrary initial guess this time will increase by 2 or 3. (A good guess can be supplied, for example, by the Appleton–Hartree equation.) This timing is 50–100 times faster than in other codes. The timing for $n_{\|} \ll 1$, which is the physically significant case, is not much longer than in the $n_{\|} = 0$ case. Otherwise more mesh points are needed to integrate over the whole $\tilde{p}$ range as indicated above.

3. *Accuracy*

The number of mesh points needed depends on the temperature and wave frequency. At 50 keV and not close to the resonance we used a 16×16 point mesh. Refining it to 32×32 points changed the results by less than 0.5%. At 300 keV we needed 20×20 points, and similarily for very low temperatures. At the very high end of the temperature range 32 points were needed in the ω^* direction.

4. *General features*

The resonances are clearly visible at the harmonics of the cyclotron frequency. The higher harmonics become more pronounced as $k_{\perp}$ increases (Fig. 3). At low temperatures the resonances are very sharp. Interestingly, the relativistic effect is very important here, because it broadens the resonances that would have become singularly sharp as $T \to 0$ in a non-relativistic theory. The resonances gradually

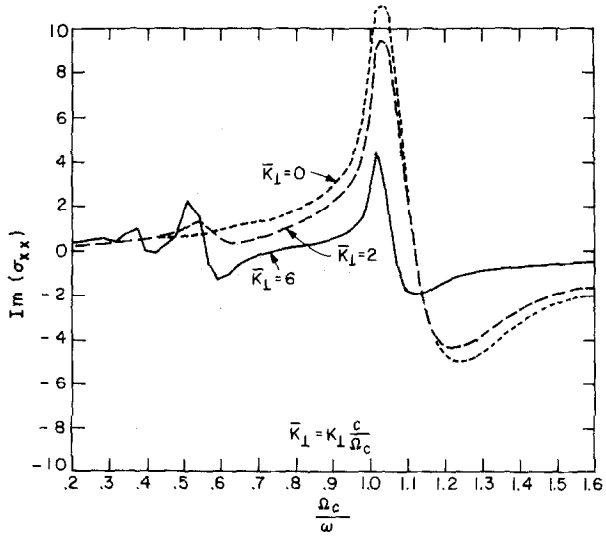

FIG. 3. $k_\perp$ dependence of Im $\sigma_{xx}(\Omega_c/\omega)$ at $T = 25$ keV.

widen and overlap as the temperature increases (Figs. 4, 5), until the curve flattens out almost completely at high temperatures, except for a development of a resonance near the velocity of light (i.e., when $kc/\omega \approx 1$) (Fig. 6). This resonance sharpens with further temperature increase (Fig. 6).

The code is available for public use, as a FORTRAN subroutine on the national Magnetic Fusion Energy computer network.

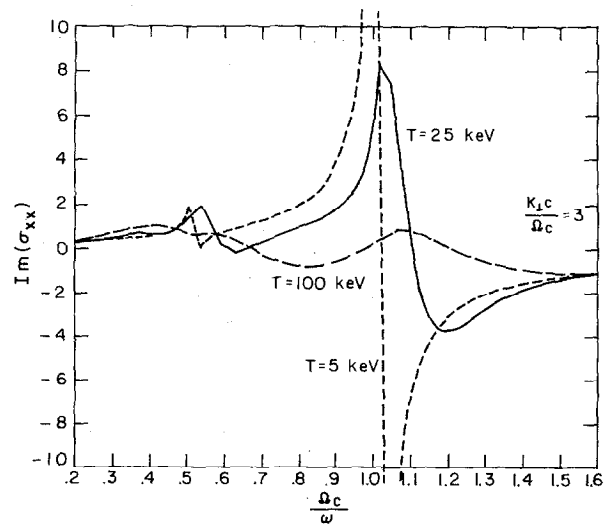

FIG. 4. Temperature dependence of Im $\sigma_{xx}(\Omega_c/\omega)$.

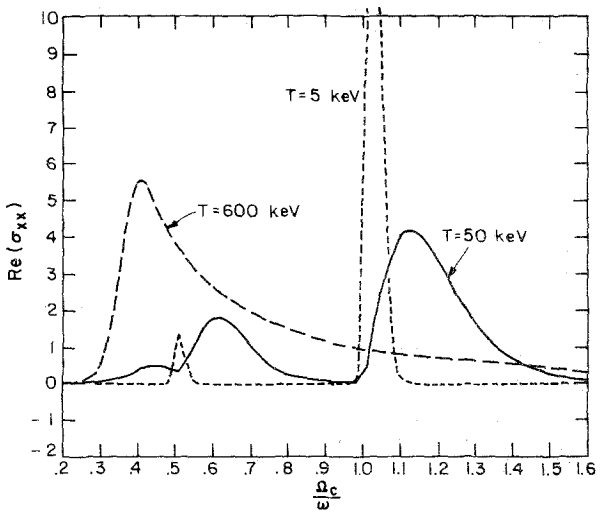

FIG. 5. Temperature dependence of Re $\sigma_{xx}(\Omega_c/\omega)$ with $k_\perp c/\Omega_c = 3$.

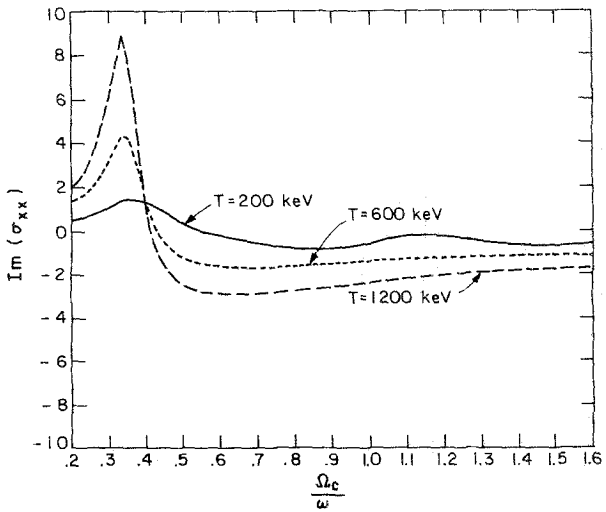

FIG. 6. Temperature dependence of Im $\sigma_{xx}(\Omega_c/\omega)$ with $k_\perp c/\Omega_c = 3$.

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